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Analogy and Classroom Mathematics Learning

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A young child sits down with blocks to solve a new problem the teacher has given her as a follow-up to earlier instruction on addition. The child exclaims: "Oh, I can do this one, this is sort of like that problem we did before."

This child's simple statement reflects a sophisticated recognition of analogical similarity between the mathematical structure of two instances, separated by time and context. Supporting the flexible, generative understanding reflected in this child's analogy lies at the heart of high quality mathematics instruction. The domain structure of mathematics creates an epistemology of necessary classroom mathematical knowledge that is quite different from retention of verbatim details, as might be privileged in other academic domains such as geography or spelling. In fact, information taught in mathematics classrooms is rarely instructed with the intention that children retain the verbatim details (e.g., the context or numbers used in problem 4). Rather, mathematical proficiency is more directly related to learners' ability to draw inferences from prior knowledge and instruction to represent and solve previously unseen problems (National Research Council [NRC], 2001; National Mathematics Advisory Panel [NMAP], 2008).

Mathematics is a system for rule-based manipulation of numbers, or "anything that plays by the rules" (Gallistel & Gelman, 2005), that is accessible to even very young children (Gelman

& Gallistel, 1978/1986). The rules themselves combine into structured systems that can be instantiated in widely varied representations. Once the structured systems have been instantiated into varied representations, however, recognizing their similarity is not a trivial cognitive act. Varied representations may include multiple mathematical problems, abstract concepts and a problem context, graphical or physical manipulatives as representations. Some of these representations appear quite similar at a surface level, using similar sized numbers and mathematical form (e.g., "3 + 4 = ?" and "5 + 3 = ?"), while others appear different at a surface level (e.g., an equation and a word problem with the same mathematical composition). Many novice learners are misled by surface, or featural characteristics of mathematical representations, and tend to either fail to notice commonalities between representations, or draw false parallels between them (e.g. using the same procedure to solve two mathematically different problems about trains).

In spite of the difficulties, recognizing commonalities in mathematical structure across contexts is a critical skill, and is a key element of mathematical proficiency (NRC 2001; NMAP, 2008). The ability to notice commonalities between representations allows learners to build on prior instruction to solve new homework or test problems, as well as draw more sophisticated connections between concepts. While much research on transfer and generalization points to the challenges of fostering this ability, either as a general reasoning skill or within particular content areas, the cognitive underpinnings of relational reasoning are less frequently discussed in the educational literature. Drawing from basic cognitive research on analogical reasoning and development allows for new insights into strategies for teaching analogical thinking in mathematics.

This chapter reviews a line of research on analogy that draws from basic studies of children's cognition and observations of classroom practices of analogy to generate classroom-feasible pedagogical practice recommendations. Analogy is first defined and its relations to classroom mathematics proficiency are discussed. Basic research on analogical reasoning and problem solving in adults and children is next reviewed. Third, an international study of mathematics teaching by analogy is described, in which teaching practices were examined in light of the basic research. The analysis led to practice recommendations that derive from everyday teaching in the U.S. and two higher achieving countries, China (Hong Kong) and Japan. Finally, controlled experiments are reported in which these recommendations were tested and shown to positively impact learners' mathematical proficiency in instructed topics. Overall, this chapter argues that U.S. mathematics teachers' practices of analogy need strengthening, and that doing so by adding elaborative cues could have broad implications for improving children's mathematical proficiency.

Defining Analogical Reasoning

Analogical reasoning may be a uniquely human capacity that is central to complex reasoning and learning (Gentner, 2003). While many people associate the term *analogy* with the form "a" is to "b" as "c" is to "d", the cognitive skill is widely recognized to be a much more integral part of the way humans process our environment. Infants attend to relations very early (e.g., Baillargeon & Hanko-Summers, 1990), and show problem solving by analogy in the first year of life (Chen, Sanchez & Campbell, 1997). This skill seems to provide a bootstrapping function, enabling children to draw on their prior knowledge to comprehend and reason about novel and increasingly complex environments (Gentner, 2003).

Definitions of analogical reasoning have taken several forms. Gentner (1983) proposed the structure-mapping model of analogy as the process of matching system-wide correspondences between the structured relations that comprise two or more entities. Thus the system of relations within one analog (e.g., a hen and a chick) is recognized as corresponding to the system of relations within another analog (e.g., a mare and a foal). Individual elements within the systems then can be aligned and mapped together (e.g., the hen is like the mare).

An important element of this definition is the distinction between similarity based upon relational correspondences (e.g., a maternal relationship), and object correspondences (e.g., hens do not look like horses). Analogies may be formed between two structures that share no surface features, or those that share both surface and structural similarity (Gentner, 2003). Analogies are therefore partial similarities between different situations that support further inferences. These may be asymmetrical systems, such that the base is better known than the target, or they may be equally well known. An analogy may result in novel inferences about the target, or about the commonalities or differences between the representations.

Holyoak and colleagues have taken a related position, though they have focused on the role of pragmatics. Specifically, they consider the ways in which context and reasoners' goals impact source analog retrieval and structure mapping (see Holyoak & Thagard, 1995; Spellman & Holyoak, 1996). Holyoak & Thagard (1995) proposed the multi-constraint theory of analogy, positing that reasoners settle upon particular correspondences based on their goals for the analogy. In the math domain, for example, a mathematics teacher might develop a different relational mapping between two problems when attempting to show students how to find a solution, versus when seeking to help her students to better understand the common conceptual structure.

Defining analogy for consideration in the mathematics classroom context is best accomplished through a combination of these approaches. For the remainder of the chapter, analogy is treated as a goal directed cognitive act of aligning and mapping relational correspondences between structural systems. We turn next to the relationship between analogical reasoning and classroom mathematical knowledge.

Relational Thinking and Mathematical Proficiency

Mathematical proficiency, as defined by the NRC (2001, p. 16) and recently endorsed by the NMAP (2008), involves five strands. These are 1) conceptual understanding, 2) procedural fluency, 3) strategic competence, 4) adaptive reasoning, and 5) productive disposition. Applying the analytical lens of analogical reasoning reveals that at least the first, second, and fourth of these strands, as articulated by the NRC, clearly engage relational thinking. The implications of relational thinking for these three aspects of mathematical proficiency are discussed briefly.

In the conceptual understanding strand, students must understand relations between concepts and operations. The ability to integrate new rules into learners' larger, stored relational structures relies on drawing structural correspondences between previous and new instruction. Deeply integrated knowledge provides a foundation for conceptual understanding.

In the procedural fluency strand, students must demonstrate the ability to use procedures appropriately, which often requires identifying structural relations between novel problems and previously solved (or instructed) problems. The authors also note that procedural fluency involves comparatively analyzing the similarities and differences between problem features.

The strategic competence strand includes the ability to represent mathematical problems based on conceptual structure rather than on surface features. Relationally speaking, this can be considered as the importance of differentiating between object features and mathematical structure. Deep attention to structure should help students recognize that changes in surface feature do not alter solution strategies.

In a broader way, relational reasoning lies at the heart of the NRC's (2000) multi-strand definition of mathematical proficiency in which they argue that deep understanding and productive problem solving requires that learners connect mathematical knowledge across these multiple strands (p. 118). While both teachers and educational researchers largely agree on the goal to lead students to develop richly connected knowledge, designing and implementing such instruction is challenging and often less than successful (Hiebert et al, 2003). This chapter next reviews a cognitive perspective on factors that facilitate or constrain comparative thinking.

Analogical Reasoning in Problem Solving and Learning

Much basic research indicates that analogy is a fundamental part of the way children and adults reason about their world. Despite famous cases of analogy use in scientific discoveries, most problem solving by analogy happens within mundane everyday reasoning, and involves smaller leaps of inference. Children learn to solve problems by analogy within the first year of life (Chen et al., 1997), and analogies are a regular part of classroom mathematical discourse (see English, 1997). Learners are also quite good at structure mapping between source and target representations when they are aware that they should be doing so (e.g., Gick & Holyoak, 1980; Brenner et al, 1997; Novick & Bassok, 2005).

Learning from Structure-Mapping

Analogical reasoning can facilitate problem solving, inferential thinking, and learning new strategies as long as participants are provided with key support (e.g., see Brenner et al, 1997; Chen & Klahr, 2008; Novick & Bassok, 2005; Rittle-Johnson & Star, 2007). In a mathematics study that illustrates this potential, Novick & Holyoak (1991) provided participants with a problem and solution, and then evaluated their later performance on an analogous test problem when given one of three types of hints with varying levels of specificity, or no hint.

All hints led to initially more analogical transfer than no hints, and there was a direct correlation between the specificity of the hint and participants' likelihood of noticing and effectively using the source analog as a base for the analogy. The more specific the hint, the better the likelihood that participants performed analogical transfer. Importantly, those who were successful later showed enhanced transfer rates on delayed final problems when solving them without any cues or hints. These data suggested that learning by doing analogical reasoning, even with high support by an instructor such as a very explicit hint, may lead to increasingly schematized, generalizable knowledge representations. The data also indicate that the nature of cues supporting instructional analogies may crucially impact learning.

Rittle-Johnson and Star (2007) have recently shown similar success with facilitating middle-school students' comparisons between two accurate solution strategies to a single problem. Such comparisons led to higher performance on measures of retention as well as on measures of conceptual, schematized understanding. Providing learners with the same information in serial order, on different pages of a packet, did not produce the same benefits.

Instructional Comparisons are Risky

Despite the evidence that analogies can facilitate problem solving both directly and through schema induction, providing an analogical reasoning opportunity to reasoners is not enough to guarantee learning or transfer. Rates of spontaneous usage of analogies are remarkably low in experimental contexts (e.g., Gick & Holyoak, 1980; Reed, 1989). While this may under-represent the reasoning that is performed in everyday contexts in which reasoners have more expertise, classroom-learning contexts are akin to laboratory contexts in which reasoners are relative novices. Retrieval searches for relevant source analogs are closely tied to one's knowledge base. Novices are more likely to conduct a search of stored potential analogs on the basis of surface features of the test problem, while experts are more likely to search on the basis of relational structure (see Chi & Ohlsson, 2005). As a consequence, novices who have not received sufficient training to view problems more like experts and notice the key structural elements may fail to notice the relevance of a stored problem (see Novick & Bassok, 2005).

Further, instructional analogies that are not well defined can lead to overextensions or misconceptions (Zook & DiVesta, 1991). Because analogies are not isomorphs, there are always both similarities and differences between the representations. Thus, learners must receive strong scaffolding to ensure they are making valid inferences based on the structure mapping, rather than being misled by surface or irrelevant source characteristics.

Processing Demands on Analogy

Some of the difficulty and potential for missteps from analogical reasoning may be attributable to the high processing demands of representing and manipulating complex relational structures. These demands are enhanced for novices whose grasp of the relevant representations is weaker. Dual task and cognitive neuropsychological methodologies have produced evidence that working memory and executive function are critically involved in two aspects of analogical reasoning: representing and integrating relevant relations (Relational Integration), and controlling attention to competitive, irrelevant information (Interference Resolution).

Relational integration refers to the number of relations that must be held active simultaneously in order to process a complex analogy, and Halford and colleagues have hypothesized that processing demand increases as the number of relations to be integrated increases (Halford, 1993; Halford, Wilson & Phillips, 1998). Interference resolution refers to the ability to control attention and inhibit activated but irrelevant, or misleading, features of source and target analogs (e.g., attempting to map between two mathematically dissimilar word problems about trains). Experimental tasks requiring both interference resolution and relational integration showed that these demands share competitive cognitive resources. Increasing either kind of demand when both were required raised undergraduates' reaction times (Cho, Holyoak & Cannon, 2007).

Learning from analogy in instructional contexts may present even more of a cognitive challenge since resources for controlling attention and manipulating information in working memory are already taxed by lack of background knowledge. Further, children are well known to have more limited working memory and executive function resources than adults. The relations between such processing considerations and children's development of analogical reasoning are next discussed.

Development of Analogical Reasoning

While early Piagetian work on analogy suggested that analogical, higher-order reasoning was not available to children until at least early adolescence, the past two decades have revealed substantial evidence that children's analogical reasoning emerges in early childhood (see Goswami, 2001). Thus, capitalizing on children's relational reasoning capacity provides a powerful resource for aiding children in building well-structured, generalizable knowledge. In the mathematics domain, early analogical reasoning ability lays the foundation for acquiring deeply conceptual knowledge and high mathematical proficiency.

In the earliest empirical documentation of analogical transfer and problem solving, Chen and colleagues (1997) designed four experiments in which ten and thirteen month old infants solved three isomorphic problems with varying levels of object similarity. Despite this early ability to reason analogically, children's relational thinking does not approximate adults' until adolescence (Halford, 1993; Richland, Morrison & Holyoak, 2006). Children's reasoning appears to differ from adults' along two dimensions. First, the rates of attending preferentially to object similarity versus relational similarity differ, and have been charted developmentally (Gentner, 1988; Gentner & Rattermann, 1991; Richland et al, 2006). Second, children's ability to process increasingly complex relations improves with time (Halford, 1993). Thus a more nuanced awareness of children's skills is necessary to best design learning environments without overtaxing children's ability.

Theories of Analogy Development

Relational Knowledge

Understanding the mechanisms underlying children's growth in analogical reasoning over time lends insight into optimal strategies for facilitating this development. Several explanatory theories have been proposed, centering either on the explanatory role of relational knowledge or processing capacity. The Relational Primacy theory (see Goswami, 2001) posited that children's ability to reason relationally is available very early, but that effectiveness improves with children's experience. In particular, knowledge of the relations and objects present in a particular reasoning context are hypothesized to increase the likelihood that a child notices relational correspondences (see Goswami, 2001). For example, understanding the relation "cut" is necessary before a child can solve the analogy: "*bread* is to *a bread slice* as *apple* is to *?*"

In a hypothesis also related to children's knowledge, Gentner (1988) and colleagues posited that while general structure mapping skills are available to young children, their reasoning in a novel context proceeds from relying upon object similarity to reasoning on the basis of relational similarity (Gentner & Rattermann 1991). Termed the "Relational Shift" hypothesis, children with less knowledge are expected to notice and draw comparisons based on object features rather than on relational features, while children with greater knowledge would preferentially attend to relations. Evidence comes from an array of stimuli including formal analogies (e.g., " *bread* is to *a bread slice* as *apple* is to *?*"). Children before the relational shift would be expected to select an object similarity match, "ball" to replace the question mark because an apple and ball are round and red. Children after the relational shift would be expected to select a "cut apple slice" because this shared the same relationship as in the source.

Background knowledge is thus clearly an important part of analogical reasoning. At the same time, while knowledge improves the likelihood that children will be able to reason about and learn from analogies, children who demonstrate the pertinent domain knowledge still fail on analogical reasoning tasks (Richland, et al., 2006). Particularly in a learning context, where domain knowledge is incomplete by definition, other mechanisms must contribute to development.

Processing Constraints

Research with adults has demonstrated the high processing loads on working memory and executive function for relational integration and interference resolution. Studies with children show that these processing constraints may also impact the developmental trajectory. Processing capacity has been proposed to constrain children's development of analogical reasoning in two ways. Halford and colleagues have focused on the role of working memory (WM) capacity, arguing that growth in WM capacity enables children to process increasingly complex analogies with age (Halford, 1993). Richland and colleagues (2006) additionally posited the role of executive function - particularly inhibitory control of attention (see Diamond, 2002).

Data from U.S. children solving scene analogy problems indicate that these cognitive capacities both have distinct roles in children's analogical reasoning development that function above and beyond the role of prerequisite domain knowledge (Richland et al, 2006). The scene analogy task separately tests the developmental effects of relational similarity and ability to control distraction from object-based similarity, and uses counterbalancing to hold domain-specific knowledge largely constant. Pairs of meaningful visual scenes were used as stimuli in which common relations were depicted using different objects (e.g., chase, drop, kiss, pull). As shown in Figure 1, one object was highlighted in a top source picture (big monkey), and children were asked to find the corresponding object in the bottom, target picture (little girl).

Four counterbalanced versions were constructed for each of the twenty picture sets by varying two dimensions. Figure 1 shows the four versions constructed for the relation "hang." The relational shift was tested by varying the presence of a distractor - an object that appeared very similar to the highlighted source object within the target picture (Distractor condition; monkey in the bottom picture of Figure 1B, D). Second, children's ability to handle relational complexity was tested by varying the number of instances of the relevant relations within a scene that needed to be mapped (One Relation [1-R] or two Relations [2-R]). In Figure 1, the 1-R problems contained the single relation *hang from* (baby monkey, adult monkey) with the elephant as an independent entity (Figure 1A, B). In the 2-R problems the elephant was engaged to depict the two-part relational structure: *hang from* (baby monkey, adult monkey, elephant) (Figure 1C, D).

Richland et al. (2006) tested the scene analogy problems with children ages three to fourteen. In a knowledge check of the materials, children in the youngest age group (three and four years) showed over a 90% accuracy in identifying the relevant relations. This meant that any developmental differences could not be attributed to a lack of prerequisite knowledge.

Across varied instructions, the youngest children (3-4 years) always showed above chance performance, demonstrating basic structure-mapping skills and requisite knowledge of the relations. Importantly, however, their performance was significantly impacted by moving from a binary to a ternary level of relational complexity, and by adding a featural similarity distractor. Similar but less strong effects were demonstrated for 6-7 year olds, with both effects lessoning with age. By 9-11 years of age both effects were minimal, though 13-14 year olds in one sample showed a significant effect of relational complexity with these materials.

Thus in spite of prerequisite knowledge of the tested relations, children's analogical processing varied along the same dimensions identified in more complex tasks to constrain adult and aging populations' relational reasoning. These data suggest that while children have the capacity to identify and map structure across analogs, their ability to do so is limited by available resources to integrate complex relations and control responses to irrelevant object properties.

Implications for Classroom Mathematics Teaching by Analogy

Consideration of developmental constraints is therefore crucial to harnessing the potential of instructional analogies for improving children's mathematical proficiency. Instructors must ensure analogical learning opportunities do not overtax background knowledge, adequate working memory resources, or ability to avoid distraction from surface similarity. Precisely what this means to classroom teachers, however, is not immediately evident. To make practice recommendations that were more directly relevant to the complexities of classroom teaching, subsequent studies used a cognitive lens to examine teachers' typical mathematical instructional use of comparisons and analogies with respect to the learning constraints noted above. Middle school teaching was analyzed in a U.S. (Richland, Holyoak & Stigler, 2004) and an international sample of typical U.S., Hong Kong, and Japanese lessons (Richland, Zur & Holyoak, 2007).

These data were sampled from the Third International Mathematics and Science Study (TIMSS, Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999) and the subsequent Trends in International Mathematics and Science Study (Hiebert et al., 2003). The TIMSS studies are a unique video 'survey' of typical classroom teaching both in the United States and internationally with approximately one hundred teachers videotaped in each country. Each teacher and lesson was selected as a random probability sample of all lessons taught in a given school year across the country, rather than as a more typical convenience sample. Both of the original TIMSS studies showed that countries have normative pedagogical patterns. Despite some inevitable individual differences across teachers the variance between teachers within a country was much less than the variance across countries.

One pattern identified in the original TIMSS 1999 study is important to the current discussion of analogy. In an analysis of problems that drew connections between mathematics concepts, procedures, or representations, U.S. teachers were less likely than their international peers to capitalize on these learning opportunities. Teachers in all countries, including the U.S., regularly administered such problems. However, close analyses of the ways in which these problems were solved and discussed revealed that the highest achieving countries all drew out these connections and engaged the students in making connections more frequently than U.S. teachers (Hiebert et al, 2003). In fact, this was the only systematic difference between teaching in the U.S. and all higher achieving countries.

This revealing divergence was further illuminated in a secondary analysis of the TIMSS data specifically focusing on those cases in which teachers made instructional analogies (Richland, et al., 2007). Analogy use was examined in the U.S. and two high achieving regions that did not share many commonalities in normative teaching patterns - Japan, and China (Hong

Kong). Ten lessons were randomly selected from the dataset for each country, each taught by a different teacher.

Analogies were identified using an integration of the structure-mapping and pragmatic definitions of analogy from the basic research literature and observations of the classroom practices. The result was a situated definition of analogy within a mathematics classroom context. Mainly, a comparison was identified if there were readily identifiable source and target representations that shared relational structure, and there was some evidence of drawing a comparison between these representations. Connections between representations based on surface features ("this solution looks like a mess"), were not coded because they do not tax analogical reasoning. Additionally, source and target representations each were required to function as a whole within the pragmatic goal structure of the analogy. For this reason, if the learners' goal was to graph an equation or use a solution strategy to solve a problem, neither of these situations would be coded. Although a graph and an equation are different representations, they function as two parts of a single problem goal.

After identification, each analogy was then coded in many different ways. Codes were developed to reflect teachers' common practices that aligned with the cognitive factors outlined above. Codes sought to capture frequency of instructional decisions that could be expected to reduce processing load, facilitate attention to relational structure of target problems, draw learners' attention to relations versus object features, reduce competitive interference, and encourage learners to draw on prior knowledge. As codes, these translated to (yes/no): 1) produced a visual representation of a source analog versus only a verbal one, 2) made a visual representation of the source analog visible during comparison with the target, 3) spatially aligned written representations of the source and target analogs to highlight structural commonalities, 4) used gestures that moved comparatively between the source and target analogs, 5) constructed visual imagery, and 6) used likely well known source analogs.

Achievement was clearly correlated with classroom analogy practices. While teachers in all three countries used approximately the same number of analogies per lesson (there were no significant differences), *how* they organized the instructional context differed significantly. Japanese and Chinese teachers used practices of analogy that were closely aligned with the practice recommendations outlined above. As shown in Figure 2, U.S. teachers were reliably less likely to use the coded principles than either Japanese or Hong Kong teachers.

These data thus reveal U.S. teachers regularly invoke analogies in their mathematics instruction, which could serve as potent opportunities for improving mathematical proficiency. However, there are many reasons to believe that students are not benefiting from these opportunities for relational reasoning. As reviewed in this chapter, analogies do not automatically benefit learners. In particular, analogies frequently fail learners who do not notice the relational correspondences or draw misconceptions or overextensions. Rather, certain elaborative conditions of the environment must be present. U.S. teachers' infrequent use of such supportive cues during instructional analogies is likely to reduce their efficacy.

So far, these data are suggestive and theoretically grounded, but correlational. No learning data were directly tied to the videotaped classroom lessons. The following section describes experiments that directly tested the prediction that adding instructional, elaborative cues to episodes of mathematical instructional analogy would improve relational reasoning, resulting in greater mathematical proficiency with the instructed topic.

Experimental Tests of Pedagogical Support for Instructional Analogies

Three experiments in separate mathematical content areas showed benefits for teaching by analogy, and all studies further revealed that adding instructional cues to support the analogy led to more flexible, generalizable knowledge representations. Two studies were conducted with undergraduates learning GRE concepts (Richland & McDonough, under review), and the general pattern of results was replicated in a sample of children in the fifth grade learning fraction operations (Richland, under review).

Videotaped instruction was used in all three studies to provide control over the instructional manipulations. In the first experiment of the series (Richland & McDonough, under review, Experiment 1), undergraduates were randomly assigned to one of two conditions: analogy with high support cues or analogy with low support cues. In both conditions a videotaped teacher first taught and demonstrated a solution to a permutation problem:

> "*Suppose there are five people running in a race. The winner of the race will get a gold medal, the person who comes in second will get a silver medal, and the person who comes in third will get a bronze medal. How many different orders of gold-silver-bronze winners can there be?*

The teacher next taught and demonstrated a solution to a combination problem:

> *A professor is choosing students to attend a special seminar. She has eleven students to choose from, but she only has four extra tickets available. How many different ways are there to make up the four students chosen to go to the seminar?*

Permutation and combination problems share mathematical structure with one difference. All assigned roles in combination problems are equivalent (i.e. in this problem, it doesn't matter which ticket a student receives), while order of assignment to roles in permutations is critical (i.e., winning gold is different from winning bronze). Thus mathematically, one must finish a combination problem by dividing the total number of permutations by the number of possible role arrangements (i.e. in this problem, 4!).

The two videos were approximately the same length and taught the same information. The experimental manipulation rested in the pedagogical cues provided by the teacher to support students in drawing a structural comparison between the two types of problems. The low cuing condition invoked the pedagogical form identified in the U.S. TIMSS 1999 in which a structural comparison was made possible for students but was not highly supported. The teacher demonstrated and explained the solution strategy to solve the permutation problem, then erased the board. He then stated that he would next show a related but different kind of problem, and demonstrated and explained the solution strategy to solve the combination problem. The serial sequence and immediate proximity of the problems would lead some students to compare their structure. However, the student would have to retrieve the source representation (permutation problem) while considering the target combination problem, and recognize the structural similarities and differences by aligning them in mental imagery.

In contrast, the teacher in the high cuing condition left the source problem on the board while teaching the target problem, and used explicit cues to help students align the two representations. Both problems were written on the board in a parallel way such that the structure was aligned visually. The teacher also used broad gestures to move between the two representations to draw students' attention to the paired analogs.

Two types of problems were included on the posttest. High Similarity problems matched both the mathematics of the instructed problems (permutation, combination) and the surface context (winning a race and tickets to a lecture). Misleading similarity problems cross-mapped mathematics and surface contexts, such that the permutation problem was set in the context of tickets to a lecture, and the combination problem was set in the context of winning a race. The high similarity problems assessed participants' learning of, and ability to implement, instructed strategies. The misleading similarity problems were a more nuanced assessment of flexible, conceptual understanding. These captured learners' ability to represent the target problem based on mathematical structure versus surface features, and their ability to distinguish between source and target correspondences based on structural versus surface similarities.

As evident in Figure 3, the data revealed an interaction between instructional condition and problem type. Participants in both instructional conditions benefited from the instructional analogy, showing approximately 80% accuracy on the facilitory similarity problems (baseline performance with the same population was 10%). In contrast, the high cuing condition significantly outperformed the low cuing condition on the cross-mapped, misleading similarity problems (baseline level 7%). This pattern indicates that any instructional analogy was beneficial, but that adding pedagogical cues to support learners' analogical thinking led to more flexible, conceptual knowledge representations.

The same interaction between cuing and posttest problem similarity was identified in two additional studies. The second study revealed a very similar result with undergraduates learning to solve proportion word problems through an analogy between a correct solution and a common but invalid solution - use of the linearity assumption (Richland & McDonough, under review, Experiment 2). This third study replicated the result in a classroom context with school-age children learning division of rational numbers by analogy to division of natural numbers (Richland, under review).

Overall, these data indicate a reliable finding that high quality analogies can be effective learning tools, but that including additional pedagogical support strategies maximizes their impact. When given additional cues, learners seem to have developed more conceptual, schematized representations of the instructed concepts and/or more adaptive proficiency in representing new problems.

Conclusions

In conclusion, analogies are powerful learning opportunities that can deepen and shape students' mathematical proficiency. Instruction by analogy is not straightforward, however, since limits in relevant knowledge and processing capacity increase the likelihood that learners fail to notice or benefit from analogies in teaching. Aligning instruction more closely to tested strategies for facilitating relational thinking could strengthen student learning and better capitalize on instructional analogies. These include reducing processing load, facilitating attention to relational structure of target problems, drawing learners' attention to relations versus object features, reducing competitive interference, and encouraging learners to draw on prior knowledge.

Successful change in U.S. teachers' practices of analogies is unlikely to come without a conceptual shift on the part of teachers to deeply and explicitly consider everyday analogies as a complex cognitive act on the part of their students. However, the proposed strategies derive from classroom practices and involve minimal time or resource investment. With professional development, such practices could greatly impact teachers' already common use of analogy, in turn profoundly affecting students' mathematical proficiency.

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Figure Caption Page

Figure 1. Sample stimuli for four versions of the “hang” relation problems.

Figure 2. Percent of analogies by region containing cognitive supports: A) visual and mental imagery, B) comparative gesture, C) visual alignment, D) use of a familiar source, E) source visible concurrently with target, F) source presented visually. White denotes U.S. teachers, Gray denotes Chinese teachers, Black denotes Japanese teachers. (Permission for reprinting of figure pending).

Figure 3. The effects of high versus low cuing of an instructional analogy on posttest problems with varying similarity to instructed problems.

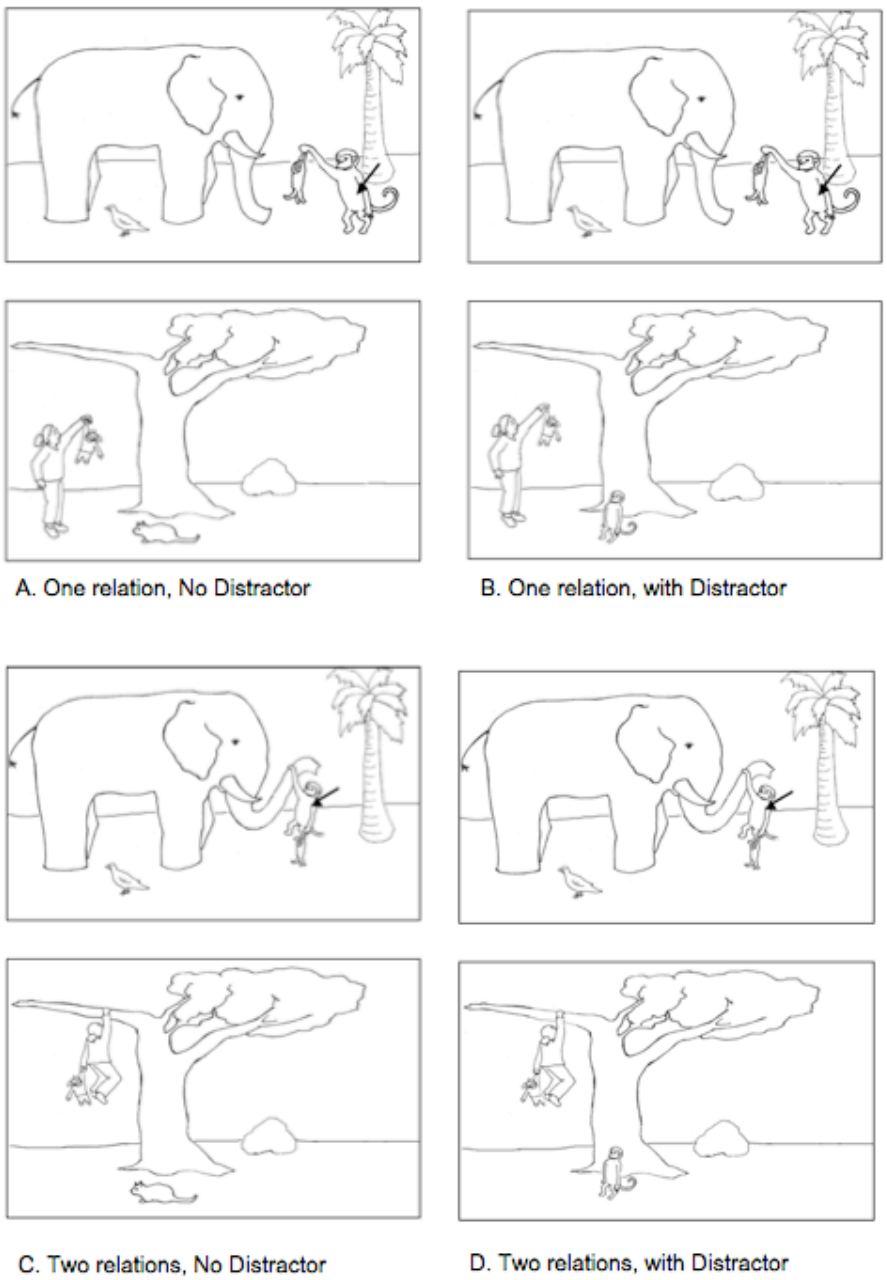
A. One relation, No Distractor
B. One relation, with Distractor
C. Two relations, No Distractor
D. Two relations, with Distractor

Figure 1

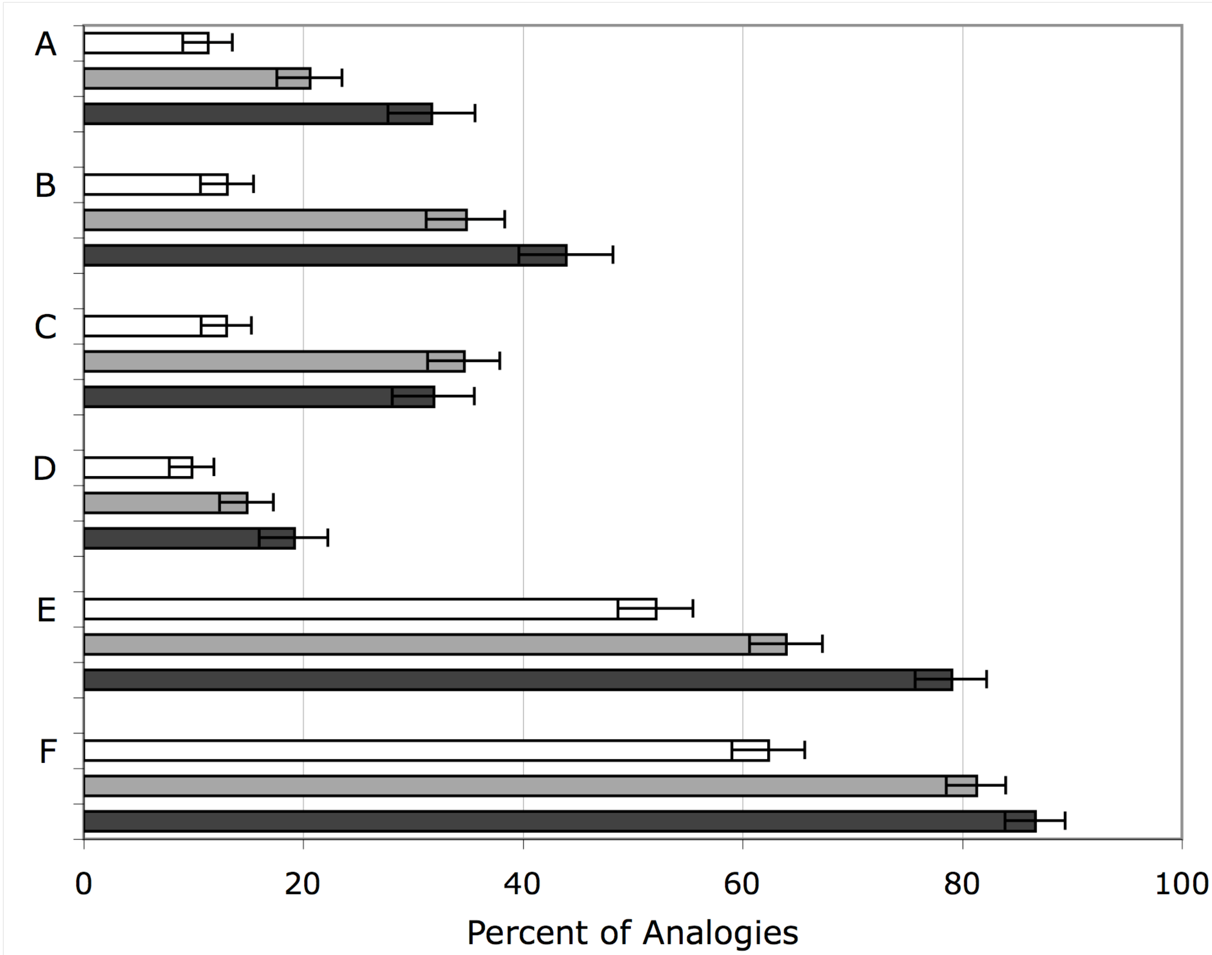

Figure 2

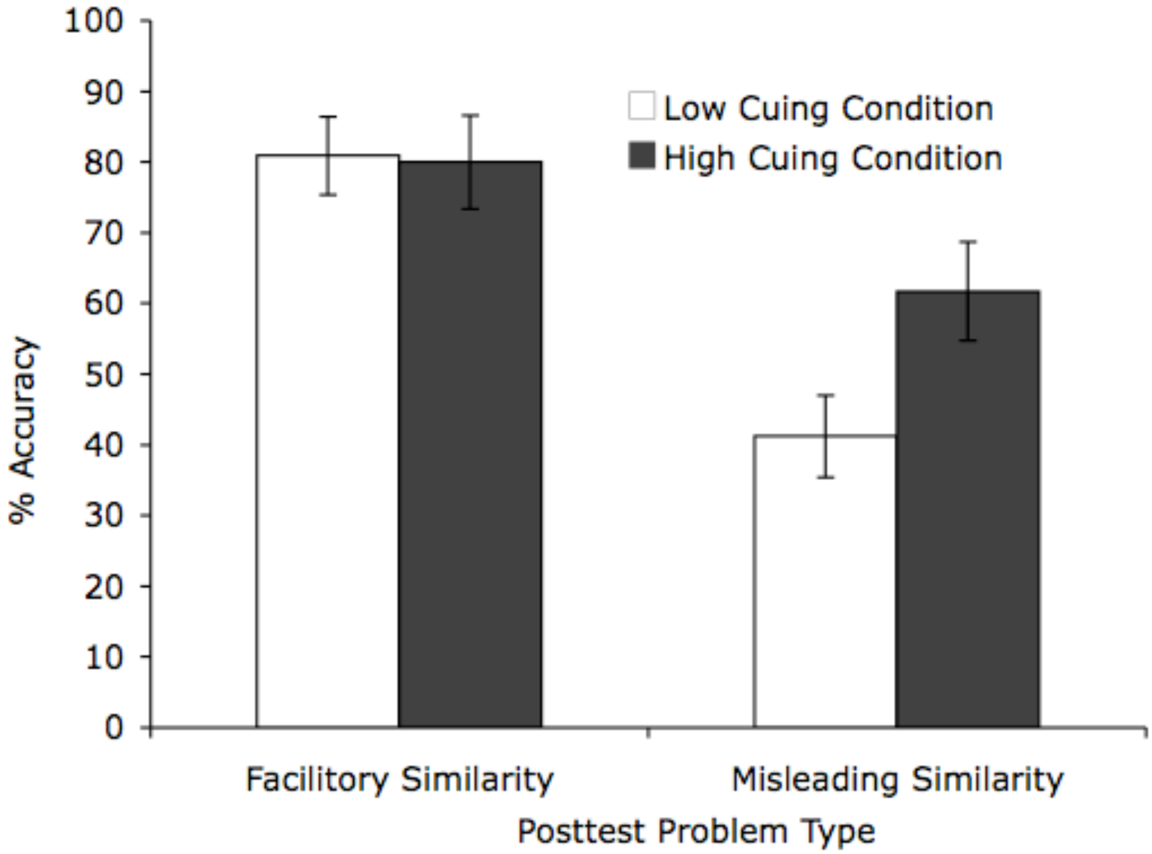
100
90
80
70
60
50
40
30
20
10
0
% Accuracy
Low Cuing Condition
High Cuing Condition
Facilitory Similarity
Misleading Similarity
Posttest Problem Type

Figure 3