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Editorial: The role of reasoning in mathematical thinking

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ABSTRACT
Research into mathematics often focuses on basic numerical and spatial intuitions, and one key property of numbers: their magnitude. The fact that mathematics is a system of complex relationships that invokes reasoning usually receives less attention. The purpose of this special issue is to highlight the intricate connections between reasoning and mathematics, and to use insights from the reasoning literature to obtain a more complete understanding of the processes that underlie mathematical cognition. The topics that are discussed range from the basic heuristics and biases to the various ways in which complex, effortful reasoning contributes to mathematical cognition, while also considering the role of individual differences in mathematics performance. These investigations are not only important at a theoretical level, but they also have broad and important practical implications, including the possibility to improve classroom practices and educational outcomes, to facilitate people's decision-making, as well as the clear and accessible communication of numerical information.

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Theorists, such as Piaget and Russell, have long discussed the inherent links between reasoning and mathematics. For example, Piaget famously claimed that children's mathematics development hinged upon their understanding of logical relations (Piaget & Cook, 1952). There is now considerable evidence that mathematics development involves magnitude processing, and numerical and spatial intuitions (e.g., Dehaene, 2011). At the same time, mathematics itself is a system of relationships that invites and invokes reasoning, and the underpinning connections between basic reasoning skills and mathematical thinking provide rich opportunities for new insights and bridging literature traditions. Reasoning skills may play a particularly important role in mathematical domains, such as algebra or geometry, relational concepts, such as

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fractions or proportions, or linking number concepts to spatial representations. Therefore, the reasoning literature offers an informative point of departure for new insights into mathematical cognition. Yet, somewhat surprisingly, reasoning and mathematics are usually studied in separate literatures and little is known about their interaction.

These investigations may have potent implications for improving our understanding of the cognitive mechanisms underlying both reasoning and mathematics skills. At the same time, they are also likely to have very practical implications, such as improving educational outcomes and people's decision-making skills, as well as communicating numerical information in an accessible way. Mathematics achievement in school is also critical for future academic and professional success, and is a topic of political and social interest in communities around the world. Therefore, there is great societal importance to identifying the reasoning skills that may contribute to effective mathematical learning in children.

The purpose of this special issue is (i) to facilitate the exchange of ideas between researchers from these two fields; (ii) to highlight the intricate connections between reasoning and mathematics that exist at multiple levels; and (iii) to leverage the reasoning literature to provide new insights into improving mathematics achievement. The issue includes a selection of nine empirical papers investigating the relationship between mathematics and reasoning. Several of these papers demonstrate how reasoning is inherently involved in mathematics. For example, not only complex numerical word problems (Primi, Donati, Chiesi, & Morsanyi, 2018), but even basic arithmetic processes are subject to heuristics and biases (Shaki, Pinhas, & Fischer, 2018; Thevenot, Fayol, & Barrouillet, 2018), and understanding ratios and fractions requires relational reasoning (Begolli et al., 2018; Gray, DeWolf, Bassok, & Holyoak, 2018, Miller Singley & Bunge, 2018; Tyumeneva et al., 2018). Processing relations is also essential for understanding transitivity and equivalence, as well as for making comparisons and ordinal judgements (Fyfe & Brown, 2018; Morsanyi, McCormack, & O'Mahony, 2018). At a more abstract level, analogical reasoning can also be used to transfer knowledge across contents, and deduction can help us to draw novel conclusions. Below we briefly review these papers by breaking them down into domains of research.

Basic arithmetic operations: heuristics and biases

Two studies in this collection examined the roles of heuristics and biases in basic arithmetic operations. Shaki et al. (2018) investigated the role of heuristics and biases in mental arithmetic. In three experiments, participants solved simple addition and subtraction problems, and were asked to present the outcome of the arithmetic problems by using a magnitude production task, where the length of a 1-unit line had to be increased bi-directionally by pressing a button. Shaki et al. (2018) demonstrated that people's estimates of the outcome of arithmetic operations were influenced by at least three competing heuristics: the *more or less heuristic* (i.e., addition and subtraction are semantically linked to more and less, respectively), the *anchoring bias* (i.e., the expectation that the first operand of subtraction problems is larger than that of addition problems if result sizes are matched), and the *sign--space association* (i.e., the learned association of addition with the right space and subtraction with the left space). These results suggest that although educated adults can rely on well-practiced algorithms when they perform arithmetic operations, similar to the domains of reasoning and decision-making, their expectations about the outcomes are influenced by a variety of heuristics and biases.

Thevenot and colleagues (2018) examined the spatial underpinnings of magnitude processing, and developed a child-friendly task to investigate preschoolers' spatial preferences when adding or removing items from objects that were organised along a line. There is a powerful general tendency among adults to represent magnitudes along an internalised left-to-right linear continuum (Dehaene, Bossini, & Giraux, 1993). It was hypothesised that if children also exhibit this tendency, then they should perform manipulations predominantly on the right end of the row of objects, and this should not be affected by whether the child is left- or right-handed. However, if numbers are not inherently associated with space, then children should predominantly perform laterality-consistent manipulations, which are easier to execute. The results showed that right-handed children exhibited a clear tendency to perform manipulations on the right end of the row, whereas left-handed children showed no clear preference. Preferences for manipulating the left or right end of the row were not significantly related to children's age, which suggests that children's preferences did not stem from cultural influences. Overall, the results suggest that laterality interacts with children's internal representations when they perform simple and concrete addition and removal operations.

Reasoning about ratios, decimals and percentages: relational reasoning and semantic alignment

A second set of papers examines relational mathematical concepts – ratios, decimals and proportions, examining ways that individuals process such relational constructs. Understanding ratios is not only essential for complex mathematics and science (Booth & Newton, 2012), but proportional reasoning is also a basic tool for understanding probabilities (e.g., Van Dooren, De Bock, Depaepe, Janssens, & Verschaffel, 2003), and, thus, inherently related to decision-making and the understanding of risks.

Miller Singley and Bunge (2018) used eye-tracking to better understand a group of highly numerate adults' spontaneous reasoning when they compared the magnitude of fractions. In particular, these researchers were interested in reliance on two strategies: a holistic strategy (i.e., within-fraction comparisons, which are aimed at extracting the magnitude of fractions and were expected to be linked to a relative dominance of vertical saccades), and a componential strategy (between-fraction comparisons, which consider the nominator and denominator separately, also taking into account the relation between the two, and were expected to be associated with horizontal saccades). Two manipulations were used in the study to elicit these strategies. The eye gaze patterns revealed that, regardless of the manipulation, participants predominantly relied on componential processing. In other words, participants' preferred strategy was to reason about relations, instead of the option to compute/estimate the magnitude of fractions. In fact, they only engaged in computations if the magnitude comparison task could not be solved by a reasoning strategy.

The paper by Gray et al. (2018) focuses on fractions, as well as their links with decimals and percentages. Conceptually, the three notations are closely related, and they are also used interchangeably in various fields, though they may appear different to a student learning them. One research question concerned how people conceptualise percentages. Percentages can be thought of as fractions with the fixed denominator of 100, but they also share the important and powerful characteristic of decimals that they have a natural ordering along a single dimension, which makes it easy to process the magnitude information that they convey. Thus, one important research question was whether people process percentages like fractions or like decimals. This question was investigated in the context of the research hypothesis that whereas decimals are conceptually linked to continuous quantities and naturally express magnitudes on a single dimension, fractions are conceptually linked to discrete, countable quantities and naturally express relations between subsets.

Gray et al. (2018) presented their participants with tasks that required magnitude comparisons and relation judgements between fractions/decimals/percentages and a visual display, which was either continuous or discrete. The results indicated that percentages were processed very much like decimals, yielding faster decisions than fractions for magnitude comparisons, but less accurate and slower decisions than fractions for relational judgements with discrete quantities. These results suggest that, rather than simply serving as notational variants, different formats of numbers are naturally well suited to represent different kinds of real-world quantities and relations between them, a process which has also been referred to as semantic alignment (Bassok, 2001).

Tyumeneva et al. (2018) have addressed the fundamental question of whether semantic alignments depend on cultural and instructional factors, or if they result from a basic understanding of mathematical representations as analogical models of real-world scenarios. Specifically, these authors tested for the presence of semantic alignments both in textbooks and in the performance of students in the Russian Federation, a country in which the math curriculum emphasises measurement (i.e., a continuous entity) when introducing rational numbers. In contrast with previous findings in the USA and South Korea (Lee, DeWolf, Bassok, & Holyoak, 2016), textbooks in Russia included continuous entities in both decimal and fraction problems. However, similar to students in the USA and South Korea, Russian students showed semantic alignments when asked to create word problems involving fractions and decimal numbers. Thus, the study suggests that semantic alignments are not driven by cultural and instructional factors, but rather they reflect students' intuitive understanding of real-world situations.

Together with the studies by Gray et al. (2018) and Miller Singley and Bunge (2018), these findings strongly support the notion that people understand ratios as a representation of relations between discrete quantities, whereas percentages and decimals naturally express magnitudes, despite their being mathematically and logically similar. These findings have important implications for the applied areas of educational design and risk communication. For example, the spontaneous association of percentages with magnitudes might explain why people often misinterpret relative risks when they are expressed in percentages (e.g., Malenka, Baron, Johansen, Wahrenberger, & Ross, 1993).

Logic, reflection and mathematics: separating the ingredients of rational thinking

Several of the papers in this special issue explore the relationships between numerical cognition and other individual differences, including other reasoning skills, gender, and executive function. Morsanyi et al. (2018) investigated the shared underlying representations of deductive reasoning and numerical cognition in educated adults. These authors have focused on two types of deductive inference: transitive reasoning (i.e., another form of relational reasoning – e.g., Waltz et al., 1999), and conditional inference. Accuracy on the transitive reasoning task was related to performance on a "number line" task. This is consistent with the long-standing idea that transitive relations may be mapped onto a linear spatial representation that is somewhat similar to the "mental number line" associated with numerical processing (Moyer & Landauer, 1967; Potts, 1974). Importantly, however, Morsanyi et al. (2018) also showed that the relationship between deductive reasoning and math skills depends on the type of task. Conditional reasoning did not appear to be related to number line placements (nor was it related to transitive reasoning). However, conditional reasoning skills were associated with individual differences in arithmetic performance, and both of these skills shared an underlying requirement to process order information. These findings provide convincing evidence that the relationship between deductive reasoning and math skills is task-dependent, a finding which is consistent with neuroimaging evidence indicating that both deductive reasoning and numerical cognition rely on a distributed and heterogeneous network of brain regions reflecting the engagement of several processes (Fias, Menon, & Szűcs, 2013; Prado, Chadha, & Booth, 2011).

Gender and anxiety are additional individual differences that play complicated roles in numerical cognition. Primi and colleagues (2018) investigated the roles of gender and anxiety in performance in a key task, the Cognitive Reflection Test (CRT; Frederick, 2005), which importantly combines numerical cognition and abstract reasoning ability (see Campitelli & Gerrans, 2014). Primi et al. (2018) investigated the source of the gender gap, which is typically observed in the CRT, with men outperforming women. After ensuring that the CRT was measuring the same underlying construct in men and women, the authors asked a large group of university students to complete the CRT, together with tests of math anxiety and math reasoning. Performance on the CRT was related to gender, math anxiety and math reasoning. However, the gender gap in performance was mediated by mathematical reasoning and math anxiety, such that there was no longer a direct effect of gender on cognitive reflection when math anxiety and math reasoning were accounted for. In other words, the gender difference only affected the numerical component of the task.

Reasoning and mathematics in the classroom

The final two papers similarly take an individual differences approach to reasoning and mathematics, but additionally begin to draw insights that bridge applications to the science of learning. These papers focus on executive functions (Begolli et al., 2018) and prior knowledge (Fyfe & Brown, 2018) as two individual differences that may impact the likelihood of successful reasoning within a mathematics learning opportunity.

Begolli and colleagues (2018) explored whether individual differences in executive functioning (EFs, as measured by working memory and inhibitory control tasks), known to be related to both reasoning (e.g., Halford, Wilson, & Phillips, 2010; Waltz et al., 1999) and mathematics (Bull, Espy, & Wiebe, 2008), predicted learning from a relational reasoning opportunity within a mathematics lesson. Most studies of EFs and mathematics have focused on correlations to overall achievement, but the current study found that, controlling for prior knowledge (likely impacted by EFs), *learning* from a reasoning opportunity was also predicted by EFs. This has implications for explaining the broad correlations between EFs and mathematics achievement, as well as potentially provides insight into strategies for improving students' learning in mathematics classroom by reducing load on EFs when possible.

In a second study, Begolli et al. (2018) conducted interviews with mathematics teachers from a range of schools, revealing that teachers are concerned about these types of patterns. In particular, these interviews suggested that teachers immediately considered whether new pedagogies would impact students in their classes differentially, suggesting that science of learning researchers must be attuned to the role of individual differences when drawing inferences from the scientific literature to make educational recommendations in order for them to be well received by educators.

Fyfe and Brown (2018) investigated the role of individual differences in prior knowledge and their relationships to the effectiveness of another instructional tool, feedback, which is ubiquitous in real-life educational settings. These authors conducted a meta-analytic review of eight experimental studies that investigated the effects of corrective, task-specific feedback on children's understanding of math equivalence. Two outcomes were considered: procedural knowledge (i.e., the ability to select and execute the correct action sequences to solve problems) and conceptual understanding (i.e., knowing that math equivalence is a relational concept with the meaning that two quantities are equal and interchangeable). The results indicated interestingly that feedback has positive effects for low-knowledge learners and negative effects for high-knowledge learners. Moreover, these effects were stronger for procedural than for conceptual outcomes. In fact, the positive effects of feedback in the case of low-knowledge learners were only significant for procedural outcomes. For high-knowledge learners, the negative effect of feedback was significant for both procedural and conceptual outcomes, with a stronger effect on procedural outcomes.

Conclusion

Overall, the papers in this special issue not only illustrate the multiple links between reasoning and mathematical skills, but also demonstrate how multi-faceted are these links. One recurring topic among the papers is the importance of relational reasoning for different aspects of mathematical thinking. Relational reasoning makes it possible to develop abstract knowledge on the basis of concrete experiences and to transfer knowledge across contexts. As a domain-general ability, relational reasoning is thus likely to play an important role in mathematical learning, and several papers of the special issue highlight its importance for the understanding of challenging mathematics concepts, such as ratio, proportions and percentages, which are inherently relational concepts. However, it is also clear that different types of relations are involved in different mathematical tasks. Because this is likely to affect processing requirements, it is also important for research to consider these different types of relations and how they may require different forms of reasoning. Another important topic highlighted by several papers is that magnitude may not be the only important property of a number. For example, expressing the same magnitude in different numerical and non-numerical formats can strongly affect people's representations of the mathematics. Finally, many of the papers highlight the role of individual differences in explaining performance and learning at the intersection of mathematics and reasoning, with important contributions identified for deductive and inferential reasoning skills, gender, anxiety, EFs, and prior knowledge. More generally, these articles together reveal that mathematics involve a complex set of skills that are likely to go much beyond the basic numerical and spatial intuitions that have been a major topic of research in the mathematical literature. We hope that readers find this special issue informative and helpful for researchers in both the reasoning and mathematical fields. We also hope that it will stimulate more research into the intersections between reasoning and mathematical skills, and may prompt more interactions between basic science and the science of learning.

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