Analogy and Higher Order Thinking: Learning Mathematics as an Example

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Abstract

Linking ideas, concepts, and disciplinary content is an underused yet effective educational strategy for fostering students' higher order thinking. A body of psychological research on analogical reasoning can inform the challenge of encouraging higher order thinking in schools. We focus in particular on the teaching of mathematics and highlight alignments between a psychologically based definition of higher order thinking and educational goals as described within U.S. mathematical practice standards. Finally, this analysis implicates policies for supporting students' higher order thinking including requiring assessments that capture these skills; disseminating assessment data meaningfully to help improve teachers, schools, or curricula; and designing professional development that draws explicit attention to these skills.

Keywords

higher order thinking, analogy, mathematics education, CCSS mathematical practice standards, drawing connections

Tweet

Making links (analogies) throughout their curriculum teaches students higher order reasoning, useful to learning math and other STEM fields.

Key Points

- Reasoning about links between and across the curriculum is at the core of higher order thinking skills, providing a framework to improve students' broader disciplinary reasoning skills.
- Students' working memory, executive function, and ability to inhibit impulses are essential to higher order thinking, meaning these resources must be adequately supported by instruction and assessment design.
- Standardized testing can use cognitive research on reasoning to determine optimal items for summative and formative assessment data on children's higher order thinking skills.
- Professional development can ensure teachers have the resources to teach for higher order thinking because this is not typical for U.S. teachers.

Introduction

Policy makers must attend to psychological data to meet the challenges of improving education. While political and educational consensus agree on producing youth with strong higher order thinking skills, accomplishing this task is more challenging than just increasing educators' motivation, requiring more than financial or accountability incentives. Greater incentives cannot lead teachers to provide better instruction on these sorts of reasoning if they do not understand what they are aiming to teach. Rather, the complexity of the cognition underlying higher order thinking, shown by the psychological research on these skills, describes the challenges of teaching them.

Educating youth who will become innovators and experts in their fields is a primary policy and educational goal in the 21st century (e.g., Obama, Strategy for American Innovation, 2009). Aims for education are shifting away from a need to help students acquire vast stores of crystallized knowledge—with much information easily accessible via technological resources—to a focus on the ability to create, innovate, critique, evaluate, and integrate the vast amount of information now available to emerging adults. While a challenging shift for the educational system, psychological research on insight, analysis, problem-solving, and expert-like thought has been underutilized in addressing these educational reforms. This article clarifies the policy relevance of attending to psychological research on children's development and cognition when encouraging change in the current educational system.

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Both education and psychological theory recommend drawing links and making inferences about relationships among ideas, concepts, principles, and other representations. Learning through making these connections can lead to more expert-like reasoning: Learners can subsequently make inferences about new information or contexts, adapt their thinking in new ways, think critically about whether insights are sensible, and make creative leaps of thought (e.g., Bransford, Brown, & Cocking, 1999; Common Core State Standards [CCSS], 2010; Gentner, Holyoak, & Kokinov, 2001; National Mathematics Panel, 2008; Next Generation Science Standards [NGSS], 2013). Cognitively, reasoning about relationships takes attention and support, but enables learners to transfer ideas from one context to another, make inferences, and think flexibly (see Gick & Holyoak, 1983; Richland & Simms, 2015; Rittle-Johnson & Star, 2007, 2011).

Psychological research has compared disciplinary experts with those who know many facts within a field but are not known as experts. The difference is that experts view information in their disciplines using integrative approaches, focusing on the relationships between ideas, problems, or systems, rather than individual facts or phenomena (for review, see Nokes, Schunn, & Chi, 2010). Although psychological research has long explored the nature of expert-like, adaptable, and flexible knowledge, little of that work has connected with educational research and policy aiming to support those very skills. This separation has had the consequence that insights from the psychological literature have not been well connected to practice, and policies for supporting children in developing these high-quality skills have not been fully informed.

Integrating Psychological Research on Reasoning With Educational Practice

In part, the disconnect between psychological research on reasoning and educational practice is due to different literatures and definitions of thinking and reasoning skills. A first step to integrating these literatures is converging on a cognitively informed definition of higher order thinking skills. This would enable research, teaching, testing, and standards to share a common framework for defining educational goals.

We provide a definition of higher order thinking grounded in cognitive research, and illustrate its alignment with educational goals using mathematics as a domain, though these ideas have relevance for science, history, and other disciplines as well (see Richland & Simms, 2015, for more discussion). The reformed mathematics and science standards provide an important guiding list for this more complex set of thinking and reasoning skills and for designing assessments that capture them within academic areas.

Then, we review key psychological research on analogical reasoning, highlighting for educators and policy makers the critical challenge of providing equitable access to higher order thinking and learning experiences. Finally, we describe specific practices for supporting students' higher order thinking, shown to be successful in supporting student learning.

Defining Analogical Reasoning and Higher Order Thinking

Analogical reasoning is a cognitive skill that underpins the conceptual process of recognizing commonalities between systems of relationships. The formal, traditional way of depicting analogy is to describe a source relationship, *"a" is to "b,"* (e.g., "*bee is to hive"*), and finding a similar relationship within a different set of objects in a target context: *"c" is to "d"* (e.g., *"bird is to nest"*). More broadly, however, analogical reasoning describes the reasoning process in which humans understand phenomena in the world as systems of relationships that may be manipulated and compared.

Analogical thought is used regularly by experts in disciplines such as science, where the aim of much scientific discovery is to understand causal or other relational systems and to build these understandings based in part on analogies to other systems (e.g., Vitruvius, 60 BC, suggested sound waves spread like water waves; Holyoak & Thagard, 1996). Increasingly, the standards for science education are echoing these goals for having students think about the physical world as systems of relationships to be compared and manipulated. Mathematics is also understood by experts as a complex system of relationships. The educational aim is for students to attain expert-like understanding of the connections between one problem, solution, or a representation and another. For example, in mathematics the quotient can be defined as the outcome of *how many times* one quantity goes into another quantity (e.g., how many times 3 goes into 12), which has a special relationship with determining the validity of repeated addition (e.g., $3 + 3 + 3 + 3$) or the relationship of two quantities (e.g., $12 / 3$). Experts recognize these relations as elements of a larger mathematical system. Thus, they flexibly draw inferences by aligning and comparing these representations, allowing them to notice the many systems of similarity within mathematics, including the relationships between $4 \times 3 = 12$ and $12 / 4 = 3$.

There is wide agreement for the need to shift children's learning away from purely content acquisition toward also training them in expert-like reasoning practices. The implications are evident in the new standards adopted in the United States for mathematics (Common Core State Standards for Mathematics [CCSSM]) and science (Next Generation Science Standards [NGSS]). At the same time, the links between these standards and cognitive research are not clearly made evident, meaning this research is unlikely to be brought to bear on instructional strategies for meeting these standards. Also, greater clarity on students' cognition could help inform the development of test items that capture the range of thought processes implied by these standards. We discuss illustrative places of overlap below.

Analogy and State Standards in Mathematics and Science

Cognitive mechanisms involved in analogical reasoning are key for understanding how children develop critical skills outlined in the current practices for mathematics and science standards. The CCSSM contains both content standards and practice standards for mathematics, with the latter being more focused on discipline-based higher order thinking skills. To clarify these relationships, we deconstruct several of the CCSSM standards for mathematical practice (MP) numbers 2, 3, 4, 5, 7, and 8, with specific attention to the way that analogical reasoning is central to these goals.

MP 2. Reason abstractly and quantitatively;
MP 3. Construct viable arguments and critique the reasoning of others;
MP 4. Model with mathematics;
MP 5. Use appropriate tools strategically;
MP 7. Look for and make use of structure;
MP 8. Look for and express regularity in repeated reasoning;
MP 2, 4, and 5 are consistent with NGSS practices (Common Core State Standards Initiative, 2010).

Reasoning abstractly (MP 2) about problems requires students to decontextualize a problem by representing it only through abstract symbols, such as numbers or shapes, and to reflect on symbols and contextualize them into their referents (Common Core State Standards Initiative, 2010). Specifically, students are expected to understand the symbols and their referents as relational elements of a mathematical structure.

From an analogy perspective, children have to first represent the relationship characterized `by each symbol. In the example above, the relation of "*bee is to hive"* was "*lives in.*" Likewise, to accomplish this mathematical standard, a child would need to recognize the commonalities between multiple representations of the same mathematical relationship. Imagine a ratio problem, such as "There are 3 pencils in a box with 9 pens. What is the ratio of pencils to pens?" constitutes multiple relations. Although a teacher may easily represent this as 3:9, this is actually a complex act of cognition that requires work on the part of a child or novice. First, the reasoner must understand the relations between the numbers 3 and 9 and their respective referents to the quantity of pencils and pens. Also, they must know the referent of the symbol ":" representing ratio.

More broadly, they should further understand that these relationships are a system that can be manipulated and represented differently. They should be aware that one would produce a different ratio for the relationship between pens and pencils. Even further, one could also represent this information as part–whole relations, such as 3/12 of the writing implements are pencils, and 9/12 are pens. Of course, these could also be written as fractions $\frac{3}{12}$, $\frac{9}{12}$ representing the part-to-whole relationships. One could go on further, but a student with high proficiency would understand the full system of relationships, with one key aspect being understanding how to flexibly manipulate the abstract relationships within ratio and proportion.

Taken together, these systems of relations comprise an internal schema, or model, from which children can build their internal representation of a mathematics concept. One way to get at this process is described in the standards as children's ability to *construct viable arguments and critique the reasoning of others* (MP 3).

Constructing viable arguments and critiquing the reasoning of others (MP 3) requires students to build on their knowledge of MP 2 by drawing valid inferences based on established assumptions, definitions, and results, or more broadly, *relations* and *schemas*. Furthermore, students have to critique the validity of others' examples. For instance, students may solve one problem using different strategies (e.g., "Lise had 3 pebbles. She gave some to Meitner and was left with 1" can be solved by counting down from 3 until reaching 1, "3 – 1 – 1 = 1" or counting up from 1 until reaching 3, "1 + 1 + 1 = 3"). Students have to evaluate the elements consisting the relations, and how relations fit within their system, in order to determine how to compare them. This is similar to evaluating multiple examples (e.g., "*bees in a hive*," "*birds in a nest*," etc.) to examine whether the same relation applies (e.g., "*lives in*"). These types of comparisons are difficult for children and novices, yet under favorable conditions, the act of evaluating and linking multiple examples provides the foundation for developing structured relations or *schemas*. These form the foundation for children to *model with mathematics* (MP 4).

Modeling with mathematics (MP 4) expects students to flexibly traverse between real-world scenarios and mathematical representations such as graphs, tables, diagrams, flowcharts, and formulas (e.g., the problem "Curie had 9 pencils. She gave some to Marie and was left with 4" represented with the formula "9 – x = 4"). This is similar to understanding the relation "*lives in*" to apply it to a real-world situation such as "*bee is to hive*." Analogical reasoning research suggests that children and novices are often misled by superficial features and regularly fail to move between the mathematical/scientific structures and real-world scenarios. For example, a division and a subtraction problem involving pizzas are misinterpreted as having the same mathematical structure. Or students might imagine a problem about speed in a race should be solved differently from the pizza problem because they do not share the same superficial features. This problem is further exacerbated because students may find it more efficient to memorize formulas without understanding the relationships they represent, often memorizing based on superficial features. Yet, modeling with mathematics requires children to draw connections between at least two systems of mathematical structures—mathematics as symbols and mathematics in a real-world scenario.

Using appropriate tools strategically (MP 5) requires first determining the key relationships within a new problem to understand which tool will be relevant for this particular set of information, such as a pencil-paper, ruler, protractor, or graphing calculator. Students who are not fully successful may grasp at particular elements of a problem to determine the match to tools, for example, keywords in a word problem such as "how long" or "show," which can not only be helpful, but also mislead the student who does not examine these words in the broader context of the relationships in the problem. For example, there is no need to use a ruler to measure the distance between two given values in a problem such as the following, but scratch paper might be helpful: "Seth's toy drives 2 feet, and Maria's drives twice as far. How far did Maria's car go?"

Looking for and making use of structure (MP 7) and *looking for and expressing regularity in repeated reasoning* (MP 8) both revolve around identifying relational structures based on functional properties of mathematics. For MP 7, students are expected to evaluate problems and examine whether known mathematical structures can be *applied* with validity. For MP 8, students are expected to examine multiple instances in the formal world in an effort to *discover* a valid structure, described in the standards as a pattern.

In MP 7, students should make use of structures such as the commutative property, $A + B = B + A$ and $A \times B = B \times A$; the associative property, $A + (B + C) = (A + B) + C$ and $A \times (B \times C) = (A \times B) \times C$; and the distributive property, $A \times (B + C) = A \times B + A \times C$. Proficient students will notice the similarity relationship between $3 \times 2 = 6$ and $3 \times (x + 1) = 6$ to discover that $x = 1$. Whereas for MP 8, students may discover mathematical structures by comparing multiple ways of arriving at the same answer, for example, different ways at reaching 32 through fractions. Some students may represent 32 as $64 \div 2$, $96 \div 3$, $128 \div 4$ while others may represent 32 as $320 \div 10$; $3{,}200 \div 100$; $32{,}000 \div 10{,}000$, and such. Through examining these relational instances, students may align their functional properties to discover a general pattern or schema, which is $nx/n = x$ for all $n > 0$.

Summary

In summary, one may identify within these MPs the role of analogical reasoning about relationships. This analysis has implications for developing a more precise definition of the type of children's thinking that the standards intend for students, getting at the core thinking—beyond a surface-level capacity to show some success in these areas (e.g., just using models or tools when explicitly told what to do with them).

This further facilitates the development of tests that align not only with the content standards but that also measure the deep thinking they are intended to capture. By considering the cognition underlying what it means to accomplish these practice standards, one may develop measures that assess not only ability to produce a specific type of answer (e.g., write a ratio) but perhaps that also measure students' ability to manipulate the information provided to show understanding of the broader system of relationships, for example, moving from ratio to proportion, or part–part to part–whole relationships.

Processing Constraints on Relational-Thinking Skills

Highlighting the relationships between analogical thought and educational aims also provides a way for the psychological research to inform teachers regarding ways that children will need particular support. Studies of analogy and relational reasoning suggest that, while these are powerful cognitive skills, they develop over time, and learners will need specific supports to accomplish them successfully. Simply providing a mathematical analogy or having a teacher demonstrate the system of relationships within a problem is not adequate for a younger child or novice to grasp these relations. As explained next, this may be because learners fail to notice the relevance of making links or lack the cognitive resources for processing those relationships. We next describe the developmental trajectory in reasoning by analogy and highlight the specific challenges that the research has identified for educators.

Noticing the Relevance of Making Links

Presenting learners with opportunities for making links and reasoning analogically can enhance their acquisition of expert-like knowledge and skills including learning, modeling, abstraction, and solution generation, but the benefits are not guaranteed. One well-established challenge to learning by analogy is that learners may not recognize the possibility or utility of making an analogy (e.g., Bransford, Brown, & Cocking, 1999; Gick & Holyoak, 1980). Thus, teachers may produce an analogy with the aim to clarify an idea or a solution method, yet students fail to notice that they are supposed to be comparing or drawing links between information.

When people identify that they should be making an analogy, they often appear competent and able to make higher order connections. However, young children or domain novices, in particular, tend to notice correspondences between based on object properties (e.g., like the appearance of a triangle, including atypical cases where the edges do not connect) rather than the relationships within those representations (e.g., the relations between angles and line segments within a polygon). Thus, in a classroom, students may be more likely to relate situations or problems on the basis of superficial, non-essential features, rather than on their key underlying relationships, whereas teachers have more content knowledge and expect the fundamental, relational commonalities will be apparent to their students.

Cross-cultural studies of teaching suggest that U.S. teachers do not always provide pedagogical supports, including visual supports and gestures, to ensure that students are looking at the key ideas, concepts, or problems to be linked (Richland, 2015; Richland, Zur, & Holyoak, 2007). Without these supports, students may fail to notice the relevance of higher order thinking despite teachers' intentions to engage them.

Cognitive Load and Ensuring Adequate Processing Resources

A second challenge is that learners may be hindered by the high processing demands of such higher order thinking (Richland & McDonough, 2010). The reasoning processes of doing analogy and finding or manipulating relationships are cognitively taxing. They require a student to hold complex representations in mind and manipulate those representations to determine alignments with solution strategies or other representations while frequently planning and executing a multi-step solution. Reasoning about relationships requires the *executive function* (EF) *system*, which is limited in capacity. With the EF system, humans hold information in mind, manipulate it, and control their attention to add only relevant information into working memory and inhibit irrelevant, distracting information (see Diamond, 2002; Miyake et al., 2000). Holding information in working memory is important for mentally coordinating and manipulating representations. To successfully make relational links, all key relationships need to be available in a reasoner's limited working-memory store while manipulating them to make inferences about their higher order relationship (Cho, Holyoak, & Cannon, 2007; Waltz et al., 2000). For example, when comparing problem solutions described by two different students, one must hold both in mind while mentally re-organizing them to determine if and how the elements correspond or whether they lead to different mathematics. The same would be true if a teacher drew a link between a prior and a new problem, such as by saying, "*Ok this next problem is like the last one, only now we'll be using a negative number for x.*"

Other aspects of EF, attentional control and inhibition, are also integral to higher order thinking by allowing a reasoner to suppress irrelevant yet potentially salient mappings (e.g., the irrelevant similarity between two solutions that both involve the same numerator in different fractions; Cho et al., 2007).

EF differs across individuals (e.g., Engle & Kane, 2004; Miyake et al., 2000), and variations in cognitive capacity may predict why some students are able to notice and benefit from opportunities for higher order thinking in the classroom. Both children and adults fail to reason relationally when under high working-memory load, when inhibitory demands are high, when under stress, or when their knowledge in a domain is limited (e.g., Gentner & Rattermann, 1991; Morrison, Doumas, & Richland, 2011; Richland, Morrison, & Holyoak, 2006). Because children's EF resources improve with age, this developmental capacity is an important consideration for teachers and test designers.

Anxiety

Finally, another factor that may predict which students fail to draw connections is anxiety. This has been examined most specifically in mathematics. *Mathematics anxiety* is fear or apprehension about doing math or math-related tasks. Math anxiety is not simply a proxy for poor math performance. Rather, people's anxiety about math—over and above their actual math ability—taxes the EF system, compromising thinking and reasoning. Underperformance may derive from engaging EF resources in distractions—verbal worry and emotion regulation efforts—reducing available resources for problem-solving and high working-memory activities (e.g., Ashcroft & Kirk, 2001; Beilock & Carr, 2005; Beilock & DeCaro, 2007). Over time, high levels of math anxiety could contribute to achievement gaps and a lack of interest in pursuing math through schooling.

Strategies for Encouraging Higher Order Thinking

Effective teaching that emphasizes higher order thinking is challenging. But, there are also opportunities to close achievement gaps and improve the quality of learning outcomes. In a large-scale data set (Crosnoe, Morrison, Burchinal, et al., 2010), exposure to teaching that included high numbers of opportunities for inferential thinking helped close achievement gaps by raising the scores of low socioeconomic status (SES), minority students who entered school low in academic skills. Although this benefit for instruction with high opportunities for inferential thinking held only if the students had a positive relationship with their teachers, the data suggest the potential for teachers to use well-supported analogy and higher order thinking opportunities to close achievement gaps.

Psychological and educational research provides insight into the conditions that support or hinder analogies in the Science, Technology, Engineering, and Mathematics (STEM) classroom (see Alfieri, Nokes-Malach, & Schunn, 2013; Rittle-Johnson & Star, 2011). This work is reviewed next.

Analogy in Mathematics Classrooms

Learning opportunities from teaching by analogies partly depend on the instructional supports teachers provide in the classroom. Successfully orchestrating a classroom lesson that helps students link ideas and make comparisons is challenging (Ball, 1993; Stein, Engle, Smith, & Hughes, 2008). This is particularly true when considering the varying factors that play into the successes or failures of instructional analogies, such as attention, cognitive load, or anxiety as just discussed. Thus, instructional supports that offload EF resources may help students focus on the relevant features of analogies, maximizing their learning.

Analyses of classroom mathematics instruction have revealed that analogy is a common part of teacher practices in many countries, including the United States, China (Hong Kong), and Japan (Richland et al., 2007). At the same time, there are key differences in how these countries' teachers provide pedagogical cues to focus students' attention to key connections and relationships of interest, rather than allowing them to focus primarily on the surface-level appearance of the phenomena (problems, models, tools, solution strategies, etc.). In parallel to achievement patterns, U.S. teachers were less likely to provide these cues than teachers in higher achieving countries—Japan and China.

Overall, U.S. teachers use analogies with close to the same frequency as teachers in higher achieving countries, but less than 1% of presented problems were identified as enabling students to draw connections as instruction unfolded (Hiebert et al., 2005). U.S. teachers typically control the analogies and engage students only in procedural aspects of the analogy, not requiring students to attend to the structural alignments between representations to participate (Richland, Holyoak, & Stigler, 2004). This may lead to failed opportunities for students to learn the deep aspects of concepts and transferable knowledge, as is typically found in laboratory studies. Further research suggests that teachers' rationale for using multiple strategies is not to afford students opportunities to compare, contrast, and think critically about these representations, but to support students' differences in *learning styles* (Lynch & Star, 2013), despite lacking evidence of learning styles (Pashler, McDaniel, Rohrer, & Bjork, 2008).

Even so, teaching mathematics with analogy can lead to gains not only in knowledge of procedures but also conceptual, flexible knowledge, when supported appropriately. This work has been conducted in mathematics classrooms (Begolli & Richland, 2015; DeCaro & Rittle-Johnson, 2012; Guo & Pang, 2011; Richland & Hansen, 2013; Rittle-Johnson & Star, 2009; Rittle-Johnson, Star, & Durkin, 2009; Schwartz, Chase, Oppezzo, & Chin, 2011; Schwartz & Martin, 2004; Star & Rittle-Johnson, 2009; Vamvakoussi & Vosniadou, 2012), and in one-on-one research with school-age children (Hattikudur & Alibali, 2010; Thompson & Opfer, 2010). Work on analogy instruction in science is also laudable (Aubusson, Harrison, Ritchie, & Fogwill, 2005; Brown & Salter, 2010; Clement, 1982, 1993; Clement & Brown, 2008; Else & Clement, 2003; Jee et al., 2013) in part highlighting the utility of analogies to bridge novel and familiar concepts (Clement, 1993; Kapon & diSessa, 2012).

Key Research-Based Strategies to Encourage Higher Order Thinking

To maximize its usability, we list key research results that provide insight into specific strategies for supporting analogical reasoning:

a. Comparing solutions of the same problem yield higher learning gains than comparing solutions of different problems with equivalent mathematical structure (Rittle-Johnson & Star, 2009);
    - Multiple solutions methods were only beneficial for students with high prior knowledge, and comparing problem types may be more beneficial for students with low prior knowledge (Rittle-Johnson et al., 2009).

b. Presenting contrasting cases is also beneficial (Schwartz et al., 2011; Schwartz & Martin, 2004);
    - Contrasting cases were shown to be beneficial when students were allowed to first explore and were given instruction after.

c. Comparing a correct and an incorrect example was more beneficial than comparing two correct examples (Durkin & Rittle-Johnson, 2012);
    - Comparing correct and incorrect examples may only be beneficial when solutions are presented simultaneously (Begolli & Richland, 2015) or provided explicit feedback (Durkin & Rittle-Johnson, 2012).

d. Presenting solution strategies simultaneously is better than presenting them sequentially;
    - While making multiple solutions to a single problem available simultaneously has led to higher learning in multiple studies, some evidence suggests that students with low prior knowledge benefit more from sequential presentations (Rittle-Johnson et al., 2009) and that solutions should be presented sequentially before being presented simultaneously (Guo & Pang, 2011).

e. Prompting students to solve problems (explore) prior to providing comparisons and explicit instruction leads to greater learning than providing instruction first (DeCaro & Rittle-Johnson, 2012; Schwartz et al., 2011; Schwartz & Martin, 2004).

f. Gestures that make links between key ideas, problems, or solutions can help students notice critical features to be compared and the utility of the comparison (Alibali & Nathan, 2007; Alibali et al., 2014; Richland, 2015).

g. Presenting a physical object that embodies the key elements of the relationship being instructed (not the appearance or features of the representation; for example, "a rubber line" can be infinitely stretched like a number line, allowing an infinite number of points between 0 and 1; Vamvakoussi & Vosniadou, 2012)

These recommendations present specific strategies that may be useful for supporting higher order thinking in everyday classroom settings. These are modes for supporting acquisition of the skills described in the CCSSM practices, and can be extended to the NGSS practices as well as other disciplines.

Summary and Policy Implications

Explicit investments in teacher professional development or research must provide teachers with better strategies for understanding and addressing these cognitively complex, higher order thinking skills within the constraints of students' limited-capacity cognitive processing systems. As laid out, the U.S. MP standards require analogical reasoning by students. While recognized as important for students, these standards are among the most challenging instructional elements for teachers. Psychological research can provide insight for teachers into the challenges and goals, as well as tools for improving the instructional context. Specific strategies can support students in making these links and connections, though the research literature is broader than could be fully reviewed here.

Moreover, policy investment in high-quality testing (either everyday or larger scale standardized tests) can provide an opportunity for supporting the development of these skills despite the often-cited discrepancy between testing and teaching higher order thinking skills. Tests can include questions that require drawing connections and making insights about the relationships between representations, problems, reading passages, historical periods, geographical contexts, and so on. Tests of higher order thinking would provide an incentive for educators and curriculum designers to ensure that these skills are central to instruction. In addition, making these results separable from student scores on basic proficiency questions would enable a national and local dialogue about the level at which students are being taught these skills.

Finally, investment in both meaningful communication of these results to stakeholders (e.g., teachers, families, students, curriculum designers, policy makers), along with professional development or curriculum reform tied to these results, could improve subsequent instructional quality. Expanding resources for communicating assessment results can enable them to play a more potent role in improving student outcomes, making these results not only summative, but also enabling them to provide a lever for change, described in the literature as formative assessment (e.g., Black, Harrison, Lee, Marshall, & Wiliam, 2004). Support could range from online or instructional strategy suggestions up to training programs with longer term feedback.

In conclusion, connecting policy and psychological aims for improvement in educating students for innovation and critical skills within academic disciplines is essential for meaningful change. Specifically, conceptualizing higher order thinking as analogy and relational reasoning provides a more cognitively specified educational target than an aim of higher level thinking broadly construed. This will better enable researchers, teachers, and test designers to align in their shared aim to improve students' broader disciplinary reasoning skills.

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